2

CHAPTER

Coulomb's Law and Electric Field Intensity

In this chapter, we introduce Coulomb's electrostatic force law and then formulate this in a general way using field theory. The tools that will be developed can be used to solve any problem in which forces between static charges are to be evaluated or to determine the electric field that is associated with any charge distribution. Initially, we will restrict the study to fields in *vacuum* or *free space*; this would apply to media such as air and other gases. Other materials are introduced in Chapter 5 and time-varying fields are introduced in Chapter 9. ■

2.1 THE EXPERIMENTAL LAW OF COULOMB

Records from at least 600 B.C. show evidence of the knowledge of static electricity. The Greeks were responsible for the term *electricity*, derived from their word for amber, and they spent many leisure hours rubbing a small piece of amber on their sleeves and observing how it would then attract pieces of fluff and stuff. However, their main interest lay in philosophy and logic, not in experimental science, and it was many centuries before the attracting effect was considered to be anything other than magic or a "life force."

Dr. Gilbert, physician to Her Majesty the Queen of England, was the first to do any true experimental work with this effect, and in 1600 he stated that glass, sulfur, amber, and other materials, which he named, would "not only draw to themselves straws and chaff, but all metals, wood, leaves, stone, earths, even water and oil."

Shortly thereafter, an officer in the French Army Engineers, Colonel Charles Coulomb, performed an elaborate series of experiments using a delicate torsion balance, invented by himself, to determine quantitatively the force exerted between two objects, each having a static charge of electricity. His published result is very similar to Newton's gravitational law (discovered about a hundred years earlier). Coulomb stated that the force between two very small objects separated in a vacuum or free space by a distance, which is large compared to their size, is proportional to the charge on each and inversely proportional to the square of the distance between them, or

$$F = k\frac{Q_1 Q_2}{R^2}$$

where Q_1 and Q_2 are the positive or negative quantities of charge, R is the separation, and k is a proportionality constant. If the International System of Units[1] (SI) is used, Q is measured in coulombs (C), R is in meters (m), and the force should be newtons (N). This will be achieved if the constant of proportionality k is written as

$$k = \frac{1}{4\pi\epsilon_0}$$

The new constant ϵ_0 is called the *permittivity of free space* and has magnitude, measured in farads per meter (F/m),

$$\epsilon_0 = 8.854 \times 10^{-12} \doteq \frac{1}{36\pi} 10^{-9} \text{ F/m} \tag{1}$$

The quantity ϵ_0 is not dimensionless, for Coulomb's law shows that it has the label $C^2/N \cdot m^2$. We will later define the farad and show that it has the dimensions $C^2/N \cdot m$; we have anticipated this definition by using the unit F/m in Eq. (1).

Coulomb's law is now

$$F = \frac{Q_1 Q_2}{4\pi\epsilon_0 R^2} \tag{2}$$

The coulomb is an extremely large unit of charge, for the smallest known quantity of charge is that of the electron (negative) or proton (positive), given in SI units as 1.602×10^{-19} C; hence a negative charge of one coulomb represents about 6×10^{18} electrons.[2] Coulomb's law shows that the force between two charges of one coulomb each, separated by one meter, is 9×10^9 N, or about one million tons. The electron has a rest mass of 9.109×10^{-31} kg and has a radius on the order of magnitude of 3.8×10^{-15} m. This does not mean that the electron is spherical, but it merely describes the size of the region in which a slowly moving electron has the greatest probability of being found. All other known charged particles, including the proton, have larger masses and larger radii, and they occupy a probabilistic volume larger than does the electron.

In order to write the vector form of (2), we need the additional fact (furnished also by Colonel Coulomb) that the force acts along the line joining the two charges and is repulsive if the charges are alike in sign or attractive if they are of opposite sign. Let the vector $\mathbf{r}_1$ locate Q_1, whereas $\mathbf{r}_2$ locates Q_2. Then the vector $\mathbf{R}_{12} = \mathbf{r}_2 - \mathbf{r}_1$ represents the directed line segment from Q_1 to Q_2, as shown in Figure 2.1. The

[1] The International System of Units (an mks system) is described in Appendix B. Abbreviations for the units are given in Table B.1. Conversions to other systems of units are given in Table B.2, while the prefixes designating powers of ten in SI appear in Table B.3.

[2] The charge and mass of an electron and other physical constants are tabulated in Table C.4 of Appendix C.

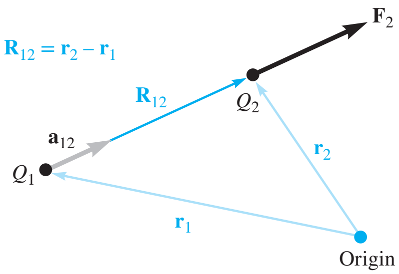

Figure 2.1 If Q_1 and Q_2 have like signs, the vector force $\mathbf{F}_2$ on Q_2 is in the same direction as the vector $\mathbf{R}_{12}$.

vector $\mathbf{F}_2$ is the force on Q_2 and is shown for the case where Q_1 and Q_2 have the same sign. The vector form of Coulomb's law is

$$\mathbf{F}_2 = \frac{Q_1 Q_2}{4\pi\epsilon_0 R_{12}^2}\mathbf{a}_{12} \tag{3}$$

where $\mathbf{a}_{12}$ = a unit vector in the direction of $\mathbf{R}_{12}$, or

$$\mathbf{a}_{12} = \frac{\mathbf{R}_{12}}{|\mathbf{R}_{12}|} = \frac{\mathbf{R}_{12}}{R_{12}} = \frac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|} \tag{4}$$

EXAMPLE 2.1

We illustrate the use of the vector form of Coulomb's law by locating a charge of $Q_1 = 3 \times 10^{-4}$ C at $M(1, 2, 3)$ and a charge of $Q_2 = -10^{-4}$ C at $N(2, 0, 5)$ in a vacuum. We want to find the force exerted on Q_2 by Q_1.

Solution. We use (3) and (4) to obtain the vector force. The vector $\mathbf{R}_{12}$ is

$$\mathbf{R}_{12} = \mathbf{r}_2 - \mathbf{r}_1 = (2-1)\mathbf{a}_x + (0-2)\mathbf{a}_y + (5-3)\mathbf{a}_z = \mathbf{a}_x - 2\mathbf{a}_y + 2\mathbf{a}_z$$

leading to $|\mathbf{R}_{12}| = 3$, and the unit vector, $\mathbf{a}_{12} = \frac{1}{3}(\mathbf{a}_x - 2\mathbf{a}_y + 2\mathbf{a}_z)$. Thus,

$$\mathbf{F}_2 = \frac{3 \times 10^{-4}(-10^{-4})}{4\pi(1/36\pi)\,10^{-9} \times 3^2}\left(\frac{\mathbf{a}_x - 2\mathbf{a}_y + 2\mathbf{a}_z}{3}\right)$$
$$= -30\left(\frac{\mathbf{a}_x - 2\mathbf{a}_y + 2\mathbf{a}_z}{3}\right) \text{N}$$

The magnitude of the force is 30 N, and the direction is specified by the unit vector, which has been left in parentheses to display the magnitude of the force. The force on Q_2 may also be considered as three component forces,

$$\mathbf{F}_2 = -10\mathbf{a}_x + 20\mathbf{a}_y - 20\mathbf{a}_z$$

The force expressed by Coulomb's law is a mutual force, for each of the two charges experiences a force of the same magnitude, although of opposite direction. We might equally well have written

$$\mathbf{F}_1 = -\mathbf{F}_2 = \frac{Q_1 Q_2}{4\pi\epsilon_0 R_{12}^2}\mathbf{a}_{21} = -\frac{Q_1 Q_2}{4\pi\epsilon_0 R_{12}^2}\mathbf{a}_{12} \tag{5}$$

Coulomb's law is linear, for if we multiply Q_1 by a factor n, the force on Q_2 is also multiplied by the same factor n. It is also true that the force on a charge in the presence of several other charges is the sum of the forces on that charge arising from each of the other charges acting alone.

D2.1. A charge $Q_A = -20\ \mu C$ is located at $A(-6, 4, 7)$, and a charge $Q_B = 50\ \mu C$ is at $B(5, 8, -2)$ in free space. If distances are given in meters, find: (*a*) $\mathbf{R}_{AB}$; (*b*) R_{AB}. Determine the vector force exerted on Q_A by Q_B if $\epsilon_0 =$ (*c*) $10^{-9}/(36\pi)$ F/m; (*d*) 8.854×10^{-12} F/m.

Ans. (*a*) $11\mathbf{a}_x + 4\mathbf{a}_y - 9\mathbf{a}_z$ m; (*b*) 14.76 m; (*c*) $30.76\mathbf{a}_x + 11.184\mathbf{a}_y - 25.16\mathbf{a}_z$ mN; (*d*) $30.72\mathbf{a}_x + 11.169\mathbf{a}_y - 25.13\mathbf{a}_z$ mN

2.2 ELECTRIC FIELD INTENSITY

Here, we introduce the first of several field quantities that we will use throughout our study. The *electric field intensity* gives the magnitude and direction of electrostatic force that would be applied to a point charge of unit magnitude that resides in the field, and as a function of its location. Emphasized here is the notion of the force acting *at a point,* and as such, the electric field intensity, like all other field quantities we will encounter, is a *point function*. Forces on larger objects, or charge distributions, must be found by summing contributions at all points that make up the object by way of a *superposition integral.* Such procedures are used in every aspect of applied electromagnetics and are introduced in later sections.

2.2.1 Electric Field Definition for a Point Charge

Consider a single point charge fixed in position, say Q_1, and move a second charge slowly around. It will be found that there exists everywhere a force on this second charge; in other words, this second charge is displaying the existence of a force *field* that is associated with charge Q_1. Call this second charge a test charge Q_t. The force on it is given by Coulomb's law, expressed by adapting Eq. (3):

$$\mathbf{F}_t = \frac{Q_1 Q_t}{4\pi\epsilon_0 R_{1t}^2}\mathbf{a}_{1t}$$

Writing this force as a force per unit charge gives the electric field intensity $\mathbf{E}_1$ arising from Q_1:

$$\mathbf{E}_1 = \frac{\mathbf{F}_t}{Q_1} = \frac{Q_1}{4\pi\epsilon_0 R_{1t}^2}\mathbf{a}_{1t} \qquad (6)$$

$\mathbf{E}_1$ is interpreted as the vector force, arising from charge Q_1, that acts on a unit positive test charge. More generally, we write the defining expression:

$$\boxed{\mathbf{E} = \frac{\mathbf{F}_t}{Q_t}} \qquad (7)$$

in which $\mathbf{E}$, a vector function, is the electric field intensity *evaluated at the test charge location* that arises from all *other* charges in the vicinity—meaning the electric field arising from the test charge itself is not included in $\mathbf{E}$.

The units of $\mathbf{E}$ would be in force per unit charge (newtons per coulomb). Again anticipating a new dimensional quantity, the *volt* (V), having the label of joules per coulomb (J/C), or newton-meters per coulomb (N · m/C), we measure electric field intensity in the practical units of volts per meter (V/m).

Most of the subscripts in (6) are now removed, reserving the right to use them again any time there is a possibility of misunderstanding. The electric field of a single point charge becomes:

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0 R^2}\mathbf{a}_R \tag{8}$$

We remember that R is the magnitude of the vector $\mathbf{R}$, the directed line segment from the point at which the point charge Q is located to the point at which $\mathbf{E}$ is desired, and $\mathbf{a}_R$ is a unit vector in the $\mathbf{R}$ direction.[3]

We arbitrarily locate Q_1 at the center of a spherical coordinate system. The unit vector $\mathbf{a}_R$ then becomes the radial unit vector $\mathbf{a}_r$, and R is r. Hence

$$\boxed{\mathbf{E} = \frac{Q_1}{4\pi\epsilon_0 r^2}\mathbf{a}_r} \tag{9}$$

The field has a single radial component, and its inverse-square-law relationship is quite obvious.

2.2.2 Fields Associated with Charges at General Locations

For a charge that is *not* at the origin of our coordinate system, the field no longer possesses spherical symmetry, and we might as well use rectangular coordinates. For a charge Q located at the source point $\mathbf{r}' = x'\mathbf{a}_x + y'\mathbf{a}_y + z'\mathbf{a}_z$, as illustrated in Figure 2.2, the field at a general point $\mathbf{r} = x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z$ can be found by expressing $\mathbf{R}$ as $\mathbf{r} - \mathbf{r}'$:

$$\begin{aligned}\mathbf{E}(\mathbf{r}) &= \frac{Q}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|^2}\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|} = \frac{Q(\mathbf{r}-\mathbf{r}')}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|^3}\\ &= \frac{Q[(x-x')\mathbf{a}_x + (y-y')\mathbf{a}_y + (z-z')\mathbf{a}_z]}{4\pi\epsilon_0[(x-x')^2+(y-y')^2+(z-z')^2]^{3/2}}\end{aligned} \tag{10}$$

Earlier, we defined a vector field as a vector function of a position vector, and this is emphasized by letting $\mathbf{E}$ be symbolized in functional notation by $\mathbf{E}(\mathbf{r})$.

Because the coulomb forces are linear, the electric field intensity arising from two point charges, Q_1 at $\mathbf{r}_1$ and Q_2 at $\mathbf{r}_2$, is the sum of the forces on Q_t caused by Q_1 and Q_2 acting alone, or

$$\mathbf{E}(\mathbf{r}) = \frac{Q_1}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}_1|^2}\mathbf{a}_1 + \frac{Q_2}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}_2|^2}\mathbf{a}_2$$

where $\mathbf{a}_1$ and $\mathbf{a}_2$ are unit vectors in the direction of $(\mathbf{r}-\mathbf{r}_1)$ and $(\mathbf{r}-\mathbf{r}_2)$, respectively. The vectors $\mathbf{r}$, $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}-\mathbf{r}_1$, $\mathbf{r}-\mathbf{r}_2$, $\mathbf{a}_1$, and $\mathbf{a}_2$ are shown in Figure 2.3.

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[3] We firmly intend to avoid confusing r and $\mathbf{a}_r$ with R and $\mathbf{a}_R$. The first two refer specifically to the spherical coordinate system, whereas R and $\mathbf{a}_R$ do not refer to any coordinate system—the choice is still available to us.

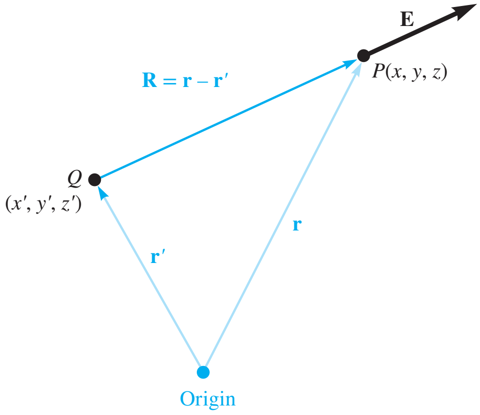

Figure 2.2 The vector $\mathbf{r}'$ locates the point charge Q, the vector $\mathbf{r}$ identifies the general point in space $P(x, y, z)$, and the vector $\mathbf{R}$ from Q to $P(x, y, z)$ is then $\mathbf{R} = \mathbf{r} - \mathbf{r}'$.

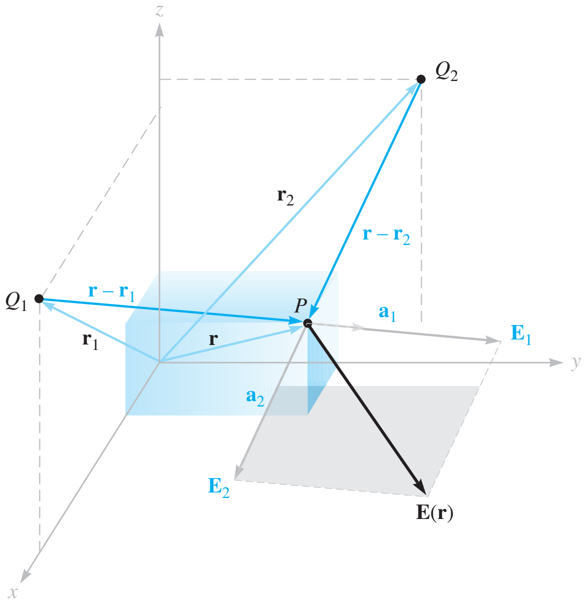

Figure 2.3 The vector addition of the total electric field intensity at P due to Q_1 and Q_2 is made possible by the linearity of Coulomb's law.

If more charges are added at other positions, the field arising from n point charges is

$$\mathbf{E}(\mathbf{r}) = \sum_{m=1}^{n} \frac{Q_m}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}_m|^2}\mathbf{a}_m \tag{11}$$

EXAMPLE 2.2

In order to illustrate the application of (11), we find **E** at $P(1, 1, 1)$ caused by four identical 3-nC (nanocoulomb) charges located at $P_1(1, 1, 0)$, $P_2(-1, 1, 0)$, $P_3(-1, -1, 0)$, and $P_4(1, -1, 0)$, as shown in Figure 2.4.

Solution. We find that $\mathbf{r} = \mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z$, $\mathbf{r}_1 = \mathbf{a}_x + \mathbf{a}_y$, and thus $\mathbf{r} - \mathbf{r}_1 = \mathbf{a}_z$. The magnitudes are: $|\mathbf{r}-\mathbf{r}_1| = 1$, $|\mathbf{r}-\mathbf{r}_2| = \sqrt{5}$, $|\mathbf{r}-\mathbf{r}_3| = 3$, and $|\mathbf{r}-\mathbf{r}_4| = \sqrt{5}$. Because $Q/4\pi\epsilon_0 = 3\times10^{-9}/(4\pi\times8.854\times10^{-12}) = 26.96\ \text{V}\cdot\text{m}$, we may now use (11) to obtain

$$\mathbf{E} = 26.96\left[\frac{\mathbf{a}_z}{1}\frac{1}{1^2} + \frac{2\mathbf{a}_x + \mathbf{a}_z}{\sqrt{5}}\frac{1}{(\sqrt{5})^2} + \frac{2\mathbf{a}_x + 2\mathbf{a}_y + \mathbf{a}_z}{3}\frac{1}{3^2} + \frac{2\mathbf{a}_y + \mathbf{a}_z}{\sqrt{5}}\frac{1}{(\sqrt{5})^2}\right]$$

or

$$\mathbf{E} = 6.82\,\mathbf{a}_x + 6.82\,\mathbf{a}_y + 32.8\,\mathbf{a}_z \ \text{V/m}$$

D2.2. A charge of $-0.3\ \mu\text{C}$ is located at $A(25, -30, 15)$ (in cm), and a second charge of $0.5\ \mu\text{C}$ is at $B(-10, 8, 12)$ cm. Find **E** at: (*a*) the origin; (*b*) $P(15, 20, 50)$ cm.

Ans. (*a*) $92.3\mathbf{a}_x - 77.6\mathbf{a}_y - 94.2\mathbf{a}_z$ kV/m; (*b*) $11.9\mathbf{a}_x - 0.519\mathbf{a}_y + 12.4\mathbf{a}_z$ kV/m

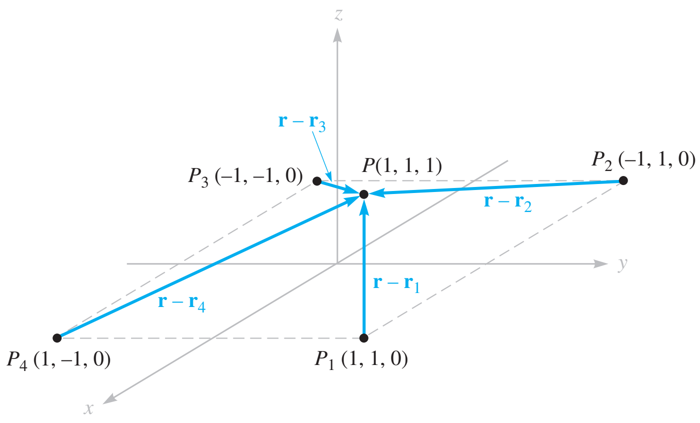

Figure 2.4 A symmetrical distribution of four identical 3-nC point charges produces a field at P, $\mathbf{E} = 6.82\mathbf{a}_x + 6.82\mathbf{a}_y + 32.8\mathbf{a}_z$ V/m.

D2.3. Evaluate the sums: (*a*) $\sum_{m=0}^{5} \frac{1+(-1)^m}{m^2+1}$; (*b*) $\sum_{m=1}^{4} \frac{(0.1)^m+1}{(4+m^2)^{1.5}}$

Ans. (*a*) 2.52; (*b*) 0.176

2.3 FIELD ARISING FROM A CONTINUOUS VOLUME CHARGE DISTRIBUTION

If we now visualize a region of space filled with a tremendous number of charges separated by minute distances, we see that we can replace this distribution of very small particles with a smooth continuous distribution described by a *volume charge density*, just as we describe water as having a density of 1 g/cm^3 (gram per cubic centimeter) even though it consists of atomic- and molecular-sized particles. This can be done only if we are uninterested in the small irregularities (or ripples) in the field as we move from electron to electron or if we care little that the mass of the water actually increases in small but finite steps as each new molecule is added.

This is really no limitation at all, because the end results for electrical engineers are almost always in terms of a current in a receiving antenna, a voltage in an electronic circuit, or a charge on a capacitor, or in general in terms of some large-scale *macroscopic* phenomenon. It is very seldom that we must know a current electron by electron.[4]

2.3.1 Volume Charge Density Definition

Volume charge density is denoted by ρ_v, having the units of coulombs per cubic meter (C/m^3).

The small amount of charge ΔQ in a small volume Δv is

$$\Delta Q = \rho_v \Delta v \tag{12}$$

and ρ_v may be defined mathematically by using a limiting process on (12),

$$\boxed{\rho_v = \lim_{\Delta v \to 0} \frac{\Delta Q}{\Delta v}} \tag{13}$$

The total charge within some finite volume is obtained by integrating throughout that volume,

$$\boxed{Q = \int_{\text{vol}} \rho_v dv} \tag{14}$$

Only one integral sign is customarily indicated, but the differential dv signifies integration throughout a volume, and hence a triple integration.

[4] A study of the noise generated by electrons in semiconductors and resistors, however, requires just such an examination of the charge through statistical analysis.

EXAMPLE 2.3

As an example of the evaluation of a volume integral, we find the total charge contained in a 2-cm length of the electron beam shown in Figure 2.5.

Solution. From the illustration, we see that the charge density is

$$\rho_v = -5 \times 10^{-6} e^{-10^5 \rho z} \text{ C/m}^2$$

The volume differential in cylindrical coordinates is given in Section 1.8; therefore,

$$Q = \int_{0.02}^{0.04} \int_{0}^{2\pi} \int_{0}^{0.01} -5 \times 10^{-6} e^{-10^5 \rho z} \rho \, d\rho \, d\phi \, dz$$

We integrate first with respect to ϕ because it is so easy,

$$Q = \int_{0.02}^{0.04} \int_{0}^{0.01} -10^{-5} \pi e^{-10^5 \rho z} \rho \, d\rho \, dz$$

and then with respect to z, because this will simplify the last integration with respect to ρ,

$$Q = \int_{0}^{0.01} \left(\frac{-10^{-5}\pi}{-10^5 \rho} e^{-10^5 \rho z} \rho \, d\rho \right)_{z=0.02}^{z=0.04}$$

$$= \int_{0}^{0.01} -10^{-5} \pi (e^{-2000\rho} - e^{-4000\rho}) \, d\rho$$

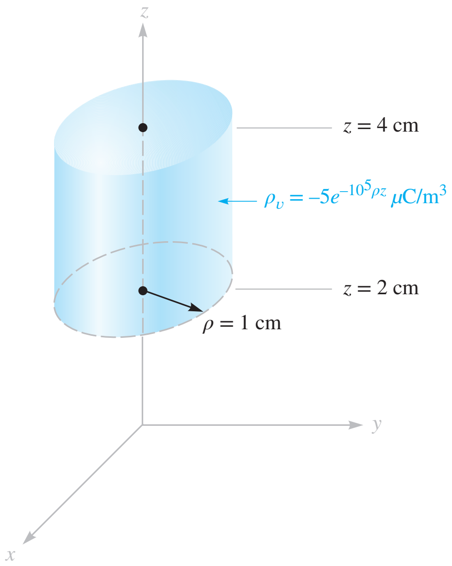

Figure 2.5 The total charge contained within the right circular cylinder may be obtained by evaluating $Q = \int_{\text{vol}} \rho_v dv$.

Finally,

$$Q = -10^{-10}\pi\left(\frac{e^{-2000\rho}}{-2000} - \frac{e^{-4000\rho}}{-4000}\right)_0^{0.01}$$

$$Q = -10^{-10}\pi\left(\frac{1}{2000} - \frac{1}{4000}\right) = \frac{-\pi}{40} = 0.0785\,\text{pC}$$

where pC indicates picocoulombs.

2.3.2 Electric Field Associated with a Volume Charge Distribution

Consider an incremental charge, ΔQ at $\mathbf{r}'$ that represents a small portion of a larger charge volume of density ρ_v, which in general may vary with position. ΔQ lies within a small volume Δv, and is thus treated as a point charge, where $\Delta Q = \rho_v \Delta v$ as before. The incremental contribution to the electric field intensity at $\mathbf{r}$ associated with this charge is written, using (10):

$$\Delta \mathbf{E}(\mathbf{r}) = \frac{\Delta Q}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|^2}\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|} = \frac{\rho_v\,\Delta v}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|^2}\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}$$

The above gives the field contribution at $\mathbf{r}$ for the small volume of charge within the larger distribution. To find the total field at $\mathbf{r}$, we sum the contributions at that point of all the charges in the distribution. This is done by first letting the volume element Δv approach zero. The effect of this is twofold: First, it provides essentially infinite spatial resolution (as the volume charge density may vary from point to point); second, the summation becomes an integral over the charge volume:

$$\mathbf{E}(\mathbf{r}) = \int_{\text{vol}} \frac{\rho_v(\mathbf{r}')\,dv'}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|^2}\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|} \tag{15}$$

This is again a triple integral, and (except in Drill Problem 2.4) we will do our best to avoid actually performing the integration.

The significance of the various quantities under the integral sign of (15) might stand a little review. The vector $\mathbf{r}$ from the origin locates the field point where $\mathbf{E}$ is being determined, whereas the vector $\mathbf{r}'$ extends from the origin to the source point where $\rho_v(\mathbf{r}')dv'$ is located. The scalar distance between the source point and the field point is $|\mathbf{r}-\mathbf{r}'|$, and the fraction $(\mathbf{r}-\mathbf{r}')/|\mathbf{r}-\mathbf{r}'|$ is a unit vector directed from source point to field point. The variables of integration are x', y', and z' in rectangular coordinates.

D2.4. Calculate the total charge within each of the indicated volumes: (*a*) $0.1 \le |x|, |y|, |z| \le 0.2$: $\rho_v = \frac{1}{x^3y^3z^3}$; (*b*) $0 \le \rho \le 0.1$, $0 \le \phi \le \pi$, $2 \le z \le 4$; $\rho_v = \rho^2 z^2 \sin 0.6\phi$; (*c*) universe: $\rho_v = e^{-2r}/r^2$.

Ans. (*a*) 0; (*b*) 1.018 mC; (*c*) 6.28 C

2.4 FIELD OF A LINE CHARGE

Up to this point we have considered two types of charge distribution, the point charge and continuous charge distributed throughout a volume with a density ρ_v C/m^3. We now consider a filamentlike distribution of volume charge density, such as a charged conductor of very small radius. It is convenient to treat the charge as a line charge of density ρ_L C/m.

Consider a straight-line charge extending along the z axis in a cylindrical coordinate system from $-\infty$ to ∞, as shown in Figure 2.6. We will find the electric field intensity **E** at any and every point resulting from a *uniform* line charge density ρ_L.

2.4.1 Setting Up the Problem: The Importance of Symmetry

Symmetry should always be considered first in order to determine two specific factors: (1) with which coordinates the field does *not* vary, and (2) which components of the field are *not* present. The answers to these questions then tell us which components are present and with which coordinates they *do* vary.

Referring to Figure 2.6, we realize that as we move around the line charge, varying ϕ while keeping ρ and z constant, the line charge appears the same from every angle. In other words, *azimuthal* symmetry is present, and no field component may vary with ϕ.

Again, if we maintain ρ and ϕ constant while moving up and down the line charge by changing z, the line charge still recedes into infinite distance in both directions and the problem is unchanged. This is *axial* symmetry and leads to fields that are not functions of z.

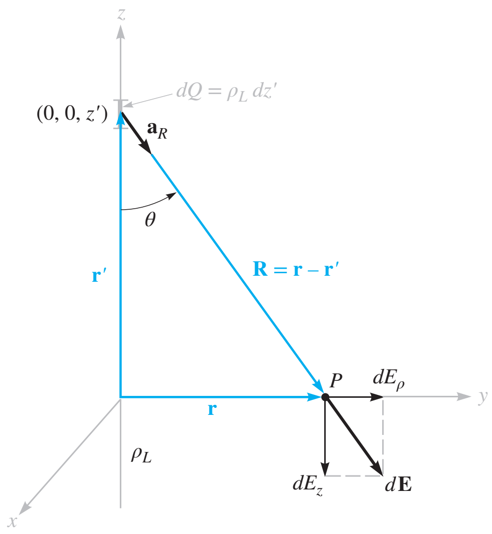

Figure 2.6 The contribution $d\mathbf{E} = dE_\rho\,\mathbf{a}_\rho + dE_z\,\mathbf{a}_z$ to the electric field intensity produced by an element of charge $dQ = \rho_L\, dz'$ located a distance z' from the origin. The linear charge density is uniform and extends along the entire z axis.

If we maintain ϕ and z constant and vary ρ, the problem changes, and Coulomb's law leads us to expect the field to become weaker as ρ increases. Hence, by a process of elimination we are led to the fact that the field varies only with ρ.

Now, which components are present? Each incremental length of line charge acts as a point charge and produces an incremental contribution to the electric field intensity that is directed away from the bit of charge (assuming a positive line charge). No element of charge produces a ϕ component of electric intensity; E_ϕ is zero. However, each element does produce an E_ρ and an E_z component, but the contribution to E_z by elements of charge that are equal distances above and below the point at which we are determining the field will cancel. Therefore only an $E\rho$ component is expected, and this will vary only with ρ. Now to find this component.

We choose a point $P(0, y, 0)$ on the y axis at which to determine the field. This is a perfectly general point in view of the lack of variation of the field with ϕ and z. Applying (10) to find the incremental field at P due to the incremental charge $dQ = \rho_L dz'$, we have

$$d\mathbf{E} = \frac{\rho_L dz(\mathbf{r} - \mathbf{r}')}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}'|^3}$$

where $\mathbf{r}' = z'\mathbf{a}_z$ and $\mathbf{r} = y\mathbf{a}_y = \rho\mathbf{a}_\rho$. The replacement of y with ρ in the last equality arises from the symmetry, in that the field in the xy plane will vary only with distance from the origin, expressed as the more general ρ direction in cylindrical coordinates. We now have

$$\mathbf{r} - \mathbf{r}' = \rho\,\mathbf{a}_\rho - z'\,\mathbf{a}_z$$

and therefore,

$$d\mathbf{E} = \frac{\rho_L dz'(\rho\,\mathbf{a}_\rho - z'\,\mathbf{a}_z)}{4\pi\epsilon_0(\rho^2 + z'^2)^{3/2}}$$

The differential field contributions to a point in the xy plane are now summed by integrating the preceding differential field over the infinite line charge:

$$\mathbf{E}_\rho = \int_{-\infty}^{\infty} \frac{\rho_L dz'(\rho\,\mathbf{a}_\rho - z'\,\mathbf{a}_z)}{4\pi\epsilon_0(\rho^2 + z'^2)^{3/2}}$$

At this point we note that the second term in the integral, involving $z'\mathbf{a}_{z'}$, integrates to zero because it gives equal and opposite contributions that cancel each other as z' changes sign at the origin. This is an example of a function that exhibits *odd parity*. This result demonstrates mathematically what was already discussed, that the z contributions to the field from the symmetric charge will cancel out. The remaining part of the integral, involving $\mathbf{a}_\rho$, is evaluated by integral tables or by a change of variable, $z' = \rho\cot\theta$, leading to:

$$E_\rho = \frac{\rho_L}{4\pi\epsilon_0}\rho\left(\frac{1}{\rho^2}\frac{z'}{\sqrt{\rho^2 + z'^2}}\right)_{-\infty}^{\infty}$$

so that

$$E_\rho = \frac{\rho_L}{2\pi\epsilon_0\rho}$$

or finally,

$$\mathbf{E} = \frac{\rho_L}{2\pi\epsilon_0\rho}\mathbf{a}_\rho \tag{16}$$

We note that the field falls off inversely with the distance to the charged line, as compared with the point charge, where the field decreased with the *square* of the distance. Moving 10 times as far from a point charge leads to a field only 1 percent the previous strength, but moving 10 times as far from a line charge only reduces the field to 10 percent of its former value. An analogy can be drawn with a source of illumination, for the light intensity from a point source of light also falls off inversely as the square of the distance to the source. The field of an infinitely long fluorescent tube thus decays inversely as the first power of the radial distance to the tube, and we should expect the light intensity about a finite-length tube to obey this law near the tube. As our point recedes farther and farther from a finite-length tube, however, it eventually looks like a point source, and the field obeys the inverse-square relationship.

2.4.2 Field of an Off-Axis Line Charge

Before leaving this introductory look at the field of the infinite line charge, it should be recognized that not all line charges are located along the z axis. As an example, consider an infinite line charge parallel to the z axis at $x = 6$, $y = 8$, shown in Figure 2.7. $\mathbf{E}$ is to be found at the general field point $P(x, y, z)$.

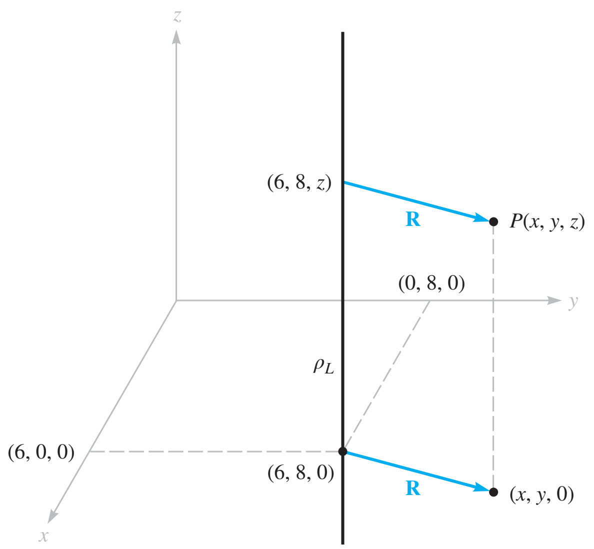

Figure 2.7 A point $P(x, y, z)$ is identified near an infinite uniform line charge located at $x = 6$, $y = 8$.

ρ is replaced in (16) by the radial distance between the line charge and point, $P, R = \sqrt{(x-6)^2 + (y-8)^2}$, and let $\mathbf{a}_\rho$ be $\mathbf{a}_R$. Thus,

$$\mathbf{E} = \frac{\rho_L}{2\pi\epsilon_0\sqrt{(x-6)^2 + (y-8)^2}}\mathbf{a}_R$$

where

$$\mathbf{a}_R = \frac{\mathbf{R}}{|\mathbf{R}|} = \frac{(x-6)\mathbf{a}_x + (y-8)\mathbf{a}_y}{\sqrt{(x-6)^2 + (y-8)^2}}$$

Therefore,

$$\mathbf{E} = \frac{\rho_L}{2\pi\epsilon_0}\frac{(x-6)\mathbf{a}_x + (y-8)\mathbf{a}_y}{(x-6)^2 + (y-8)^2}$$

We again note that the field is not a function of z.

In Section 2.6, we describe how fields may be sketched, and the field of the line charge is one example.

D2.5. Infinite uniform line charges of 5 nC/m lie along the (positive and negative) x and y axes in free space. Find **E** at: (*a*) $P_A(0, 0, 4)$; (*b*) $P_B(0, 3, 4)$.

Ans. (*a*) $45\mathbf{a}_z$ V/m; (*b*) $10.8\mathbf{a}_y + 36.9\mathbf{a}_z$ V/m

2.5 FIELD OF A SHEET OF CHARGE

Another basic charge configuration is the infinite sheet of charge having a uniform density of ρ_S C/m^2. Such a charge distribution may often be used to approximate that found on the conductors of a strip transmission line or a parallel-plate capacitor. As will be seen in Chapter 5, static charge resides on conductor surfaces and not in their interiors; for this reason, ρ_S is commonly known as *surface charge density*. The charge-distribution family now is complete—point, line, surface, and volume, or Q, ρ_L, ρ_S, and ρ_v.

2.5.1 Symmetry

Consider an infinite sheet of charge in the yz plane and again be aware of symmetry (Figure 2.8). We observe first that the field cannot vary with y or with z, and that the y and z components arising from differential elements of charge symmetrically located with respect to the point at which we evaluate the field will cancel. Therefore only E_x is present, and, as will be demonstrated, will not vary in any direction. We are again faced with a choice of many methods by which to evaluate this component, and this time we use only one method and leave the others as exercises for a quiet Sunday afternoon.

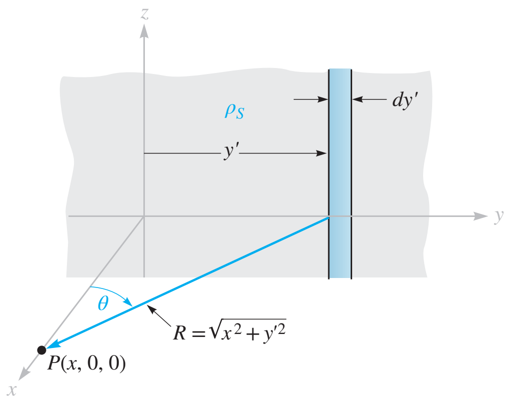

Figure 2.8 An infinite sheet of charge in the yz plane, a general point P on the x axis, and the differential-width line charge used as the element in determining the field at P by $d\mathbf{E} = \rho_S dy' \mathbf{a}_R/(2\pi\epsilon_0 R)$.

2.5.2 The Sheet Charge as an Ensemble of Line Charges

The field of the infinite line charge (16) is implemented here by dividing the infinite sheet into differential-width strips. One such strip is shown in Figure 2.8. The line charge density, or charge per unit length, is $\rho_L = \rho_S dy'$, and the distance from this line charge to our general point P on the x axis is $R = \sqrt{x^2 + y'^2}$. The contribution to E_x at P from this differential-width strip is then

$$dE_x = \frac{\rho_S dy'}{2\pi\epsilon_0 \sqrt{x^2 + y'^2}} \cos\theta = \frac{\rho_S}{2\pi\epsilon_0} \frac{x dy'}{x^2 + y'^2}$$

Adding the effects of all the strips,

$$E_x = \frac{\rho_S}{2\epsilon_0} \int_{-\infty}^{\infty} \frac{x dy'}{x^2 + y'^2} = \frac{\rho_S}{2\pi\epsilon_0} \tan^{-1}\frac{y'}{x}\Big]_{-\infty}^{\infty} = \frac{\rho_S}{2\pi\epsilon_0}$$

If the point P were chosen on the negative x axis, then

$$E_x = -\frac{\rho_S}{2\epsilon_0}$$

for the field is always directed away from the positive charge. This difficulty in sign is usually overcome by specifying a unit vector $\mathbf{a}_N$, which is normal to the sheet and directed outward, or away from it. Then

$$\boxed{\mathbf{E} = \frac{\rho_S}{2\epsilon_0}\mathbf{a}_N} \tag{17}$$

This is a startling answer, for the field is constant in magnitude and direction. It is just as strong a million miles away from the sheet as it is right off the surface. Returning to our light analogy, we see that a uniform source of light on the ceiling of a very large room leads to just as much illumination on a square foot on the floor as it does on a square foot a few inches below the ceiling. If you desire greater illumination on this subject, it will do you no good to hold the book closer to such a light source.

2.5.3 Capacitor Model

If a second infinite sheet of charge, having a *negative* charge density $-\rho_S$, is located in the plane $x = a$, the total field may be found by adding the contribution of each sheet. In the region $x > a$,

$$\mathbf{E}_+ = \frac{\rho_S}{2\epsilon_0}\mathbf{a}_x \qquad \mathbf{E}_- = -\frac{\rho_S}{2\epsilon_0}\mathbf{a}_x \qquad \mathbf{E} = \mathbf{E}_+ + \mathbf{E}_- = 0$$

and for $x < 0$,

$$\mathbf{E}_+ = -\frac{\rho_S}{2\epsilon_0}\mathbf{a}_x \qquad \mathbf{E}_- = \frac{\rho_S}{2\epsilon_0}\mathbf{a}_x \qquad \mathbf{E} = \mathbf{E}_+ + \mathbf{E}_- = 0$$

and when $0 < x < a$,

$$\mathbf{E}_+ = \frac{\rho_S}{2\epsilon_0}\mathbf{a}_x \qquad \mathbf{E}_- = \frac{\rho_S}{2\epsilon_0}\mathbf{a}_x$$

and

$$\mathbf{E} = \mathbf{E}_+ + \mathbf{E}_- = \frac{\rho_S}{\epsilon_0}\mathbf{a}_x \tag{18}$$

This is an important practical answer, for it is the field between the parallel plates of an air capacitor, provided the linear dimensions of the plates are very much greater than their separation and provided also that we are considering a point well removed from the edges. The field outside the capacitor, while not zero, as we found for the preceding ideal case, is usually negligible.

D2.6. Three infinite uniform sheets of charge are located in free space as follows: 3 nC/m^2 at $z = -4$, 6 nC/m^2 at $z = 1$, and -8 nC/m^2 at $z = 4$. Find **E** at the point: (*a*) $P_A(2, 5, -5)$; (*b*) $P_B(4, 2, -3)$; (*c*) $P_C(-1, -5, 2)$; (*d*) $P_D(-2, 4, 5)$.

Ans. (*a*) $-56.5\mathbf{a}_z$; (*b*) $283\mathbf{a}_z$; (*c*) $961\mathbf{a}_z$; (*d*) $56.5\mathbf{a}_z$ all V/m

2.6 STREAMLINES AND SKETCHES OF FIELDS

We now have vector equations for the electric field intensity resulting from several different charge configurations, and we have had little difficulty in interpreting the magnitude and direction of the field from the equations. Unfortunately, this simplicity cannot last much longer, for we have solved most of the simple cases and our new charge distributions must lead to more complicated expressions for the fields and more difficulty in visualizing the fields through the equations. However, it is true that one picture would be worth about a thousand words, if we just knew what picture to draw.

Consider the field about the line charge,

$$\mathbf{E} = \frac{\rho_L}{2\pi\epsilon_0\rho}\mathbf{a}_\rho$$

Figure 2.9*a* shows a cross-sectional view of the line charge and presents what might be our first effort at picturing the field—short line segments drawn here and there having lengths proportional to the magnitude of **E** and pointing in the direction of **E**. The figure fails to show the symmetry with respect to ϕ, so we try again in Figure 2.9*b* with a symmetrical location of the line segments. The real trouble now appears—the longest lines must be drawn in the most crowded region, and this also plagues us if we use line segments of equal length but of a thickness that is proportional to **E** (Figure 2.9*c*). Other schemes include drawing shorter lines to represent stronger fields (inherently misleading) and using intensity of color or different colors to represent stronger fields.

For the present, we will show only the *direction* of **E** by drawing continuous lines, which are everywhere tangent to **E**, from the charge. Figure 2.9*d* shows this compromise. A symmetrical distribution of lines (one every 45°) indicates azimuthal symmetry, and arrowheads are used to show direction.

These lines are usually called *streamlines*, although other terms such as flux lines and direction lines are also used. A small positive test charge placed at any point in this field and free to move would accelerate in the direction of the streamline passing through that point. If the field represented the velocity of a liquid or a gas (which, incidentally, would have to have a source at $\rho = 0$), small suspended particles in the liquid or gas would trace out the streamlines.

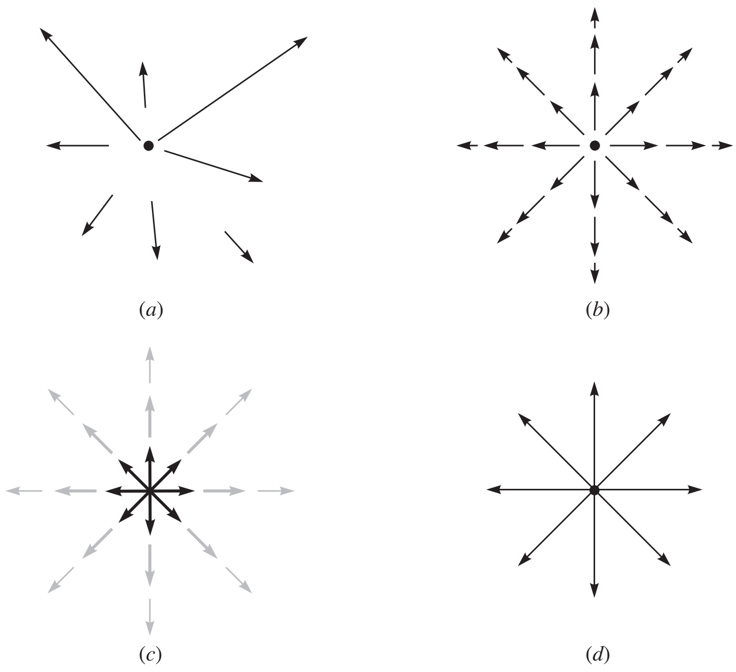

Figure 2.9 (*a*) One very poor sketch, (*b*) and (*c*) two fair sketches, and (*d*) the usual form of a streamline sketch. In the last form, the arrows show the direction of the field at every point along the line, and the spacing of the lines is inversely proportional to the strength of the field.

We will find out later that a bonus accompanies this streamline sketch, for the magnitude of the field can be shown to be inversely proportional to the spacing of the streamlines for some important special cases. The closer they are together, the stronger is the field. At that time we will also find an easier, more accurate method of making that type of streamline sketch.

If we tried to sketch the field of the point charge, the variation of the field into and away from the page would cause essentially insurmountable difficulties; for this reason sketching is usually limited to two-dimensional fields.

In the case of the two-dimensional field, we may arbitrarily set $E_z = 0$. The streamlines are thus confined to planes for which z is constant, and the sketch is the same for any such plane. Several streamlines are shown in Figure 2.10, and the E_x and E_y components are indicated at a general point. It is apparent from the geometry that

$$\boxed{\frac{E_y}{E_x} = \frac{dy}{dx}} \tag{19}$$

A knowledge of the functional form of E_x and E_y (and the ability to solve the resultant differential equation) will enable us to obtain the equations of the streamlines.

As an illustration of this method, consider the field of the uniform line charge with $\rho_L = 2\pi\epsilon_0$,

$$\mathbf{E} = \frac{1}{\rho}\mathbf{a}_\rho$$

In rectangular coordinates,

$$\mathbf{E} = \frac{x}{x^2 + y^2}\mathbf{a}_x + \frac{y}{x^2 + y^2}\mathbf{a}_y$$

Thus we form the differential equation

$$\frac{dy}{dx} = \frac{E_y}{E_x} = \frac{y}{x} \qquad \text{or} \qquad \frac{dy}{y} = \frac{dx}{x}$$

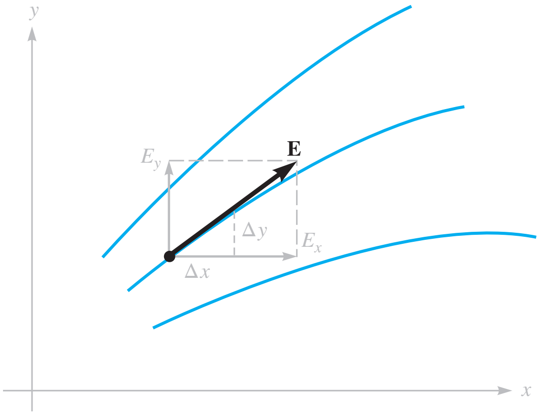

Figure 2.10 The equation of a streamline is obtained by solving the differential equation $E_y/E_x = dy/dx$.

Therefore,

$$\ln y = \ln x + C_1 \qquad \text{or} \qquad \ln y = \ln x + \ln C$$

from which the equations of the streamlines are obtained,

$$y = Cx$$

If we want to find the equation of one particular streamline, say one passing through $P(-2, 7, 10)$, we merely substitute the coordinates of that point into our equation and evaluate C. Here, $7 = C(-2)$, and $C = -3.5$, so $y = -3.5x$.

Each streamline is associated with a specific value of C, and the radial lines shown in Figure 2.9*d* are obtained when $C = 0, 1, -1$, and $1/C = 0$.

The equations of streamlines may also be obtained directly in cylindrical or spherical coordinates. A spherical coordinate example will be examined in Section 4.7.

D2.7. Find the equation of the streamline that passes through the point $P(1, 4, -2)$ in the field

$$\mathbf{E} = (a)\ \frac{-8x}{y}\mathbf{a}_x + \frac{4x^2}{y^2}\mathbf{a}_y;\ (b)\ 2e^{5x}[y(5x+1)\mathbf{a}_x + x\mathbf{a}_y].$$

Ans. (*a*) $x^2 + 2y^2 = 33$; (*b*) $y^2 = 15.7 + 0.4x - 0.08\ln(5x + 1)$

REFERENCES

1. Boast, W. B. *Vector Fields*. New York: Harper and Row, 1964. This book contains many examples and sketches of fields.
2. Della Torre, E., and Longo, C. L. *The Electromagnetic Field*. Boston: Allyn and Bacon, 1969. The authors introduce all of electromagnetic theory with a careful and rigorous development based on a single experimental law—that of Coulomb. It begins in Chapter 1.
3. Schelkunoff, S. A. *Electromagnetic Fields*. New York: Blaisdell Publishing Company, 1963. Many of the physical aspects of fields are discussed early in this text without advanced mathematics.

CHAPTER 2 PROBLEMS

2.1 Three point charges of equal magnitude q are located at $x = -2$, $y = +2$, and $y = -\sqrt{2}$. Find the coordinates of a fourth positive charge, also of magnitude q, that will yield a zero net electric field at the origin.

2.2 Point charges of 1 nC and −2 nC are located at (0, 0, 0) and (1, 1, 1), respectively, in free space. Determine the vector force acting on each charge.

2.3 Point charges of 50 nC each are located at $A(1, 0, 0)$, $B(-1, 0, 0)$, $C(0, 1, 0)$, and $D(0, -1, 0)$ in free space. Find the total force on the charge at A.

2.4 Eight identical point charges of Q C each are located at the corners of a cube of side length a, with one charge at the origin, and with the three nearest charges at $(a, 0, 0)$, $(0, a, 0)$, and $(0, 0, a)$. Find an expression for the total vector force on the charge at $P(a, a, a)$, assuming free space.

2.5 A point charge of 3 nC is located at the point (1, 1, 1) in free space. What charge must be located at (1, 3, 2) to cause the y component of **E** to be zero at the origin?

2.6 Two point charges of equal magnitude q are positioned at $z = \pm d/2$. (a) Find the electric field everywhere on the z axis; (b) find the electric field everywhere on the xy plane.

2.7 Two point charges of equal magnitude but of opposite sign are positioned with charge $+q$ at $z = +d/2$ and charge $-q$ at $z = -d/2$. The charges in this configuration form an *electric dipole*. (a) Find the electric field intensity **E** everywhere on the z axis. (b) Evaluate your part a result at the origin. (c) Find the electric field intensity everywhere on the xy plane, expressing your result as a function of radius ρ in cylindrical coordinates. (d) Evaluate your part c result at the origin. (e) Simplify your part c result for the case in which $\rho >> d$.

2.8 A crude device for measuring charge consists of two small insulating spheres of radius a, one of which is fixed in position. The other is movable along the x axis and is subject to a restraining force kx, where k is a spring constant. The uncharged spheres are centered at $x = 0$ and $x = d$, the latter fixed. If the spheres are given equal and opposite charges of Q/C, obtain the expression by which Q may be found as a function of x. Determine the maximum charge that can be measured in terms of ϵ_0, k, and d, and state the separation of the spheres then. What happens if a larger charge is applied?

2.9 A 100-nC point charge is located at $A(-1, 1, 3)$ in free space. (a) Find the locus of all points $P(x, y, z)$ at which $E_x = 500$ V/m. (b) Find y_1 if $P(0, y_1, 3)$ lies on that locus.

2.10 A configuration of point charges consists of a single charge of value $-2q$ at the origin, and two charges of value $+q$ at locations $z = -d$ and $+d$. The charges as positioned form an *electric quadrupole,* equivalent to two dipoles of opposite orientation that are separated by distance d along the z axis. (a) Find the electric field intensity **E** everywhere in the xy plane, expressing your result as a function of cylindrical radius ρ. (b) Specialize your part a result for large distances, $\rho >> d$.

2.11 A charge Q_0 located at the origin in free space produces a field for which $E_z =$ 1 kV/m at point $P(-2, 1, -1)$. (a) Find Q_0. Find **E** at $M(1, 6, 5)$ in (b) rectangular coordinates; (c) cylindrical coordinates; (d) spherical coordinates.

2.12 Electrons are in random motion in a fixed region in space. During any 1 μs interval, the probability of finding an electron in a subregion of volume 10^{-15} m^2 is 0.27. What volume charge density, appropriate for such time durations, should be assigned to that subregion?

2.13 A uniform volume charge density of 0.2 μC/m^3 is present throughout the spherical shell extending from $r = 3$ cm to $r = 5$ cm. If $\rho_v = 0$ elsewhere, find (a) the total charge present throughout the shell, and (b) r_1 if half the total charge is located in the region 3 cm $< r < r_1$.

2.14 The electron beam in a certain cathode ray tube possesses cylindrical symmetry, and the charge density is represented by $\rho_v = -0.1/(\rho^2 + 10^{-8})$ pC/m^3 for $0 < \rho < 3 \times 10^{-4}$ m, and $\rho_v = 0$ for $\rho > 3 \times 10^{-4}$ m. (*a*) Find the total charge per meter along the length of the beam. (*b*) If the electron velocity is 5×10^7 m/s, and with one ampere defined as 1 C/s, find the beam current.

2.15 A spherical volume having a 2-μm radius contains a uniform volume charge density of 10^5 C/m^3. (*a*) What total charge is enclosed in the spherical volume? (*b*) Now assume that a large region contains one of these little spheres at every corner of a cubical grid 3 mm on a side and that there is no charge between the spheres. What is the average volume charge density throughout this large region?

2.16 Within a region of free space, charge density is given as $\rho_v = \frac{\rho_0 r\cos\theta}{a}$ C/m^3, where ρ_0 and a are constants. Find the total charge lying within (*a*) the sphere, $r \leq a$; (*b*) the cone, $r \leq a$, $0 \leq \theta \leq 0.1\pi$; (*c*) the region, $r \leq a$, $0 \leq \theta \leq 0.1\pi$, $0 \leq \phi \leq 0.2\pi$.

2.17 A length d of line charge lies on the z axis in free space. The charge density on the line is $\rho_L = +\rho_0$ C/m $(0 < z < d/2)$ and $\rho_L = -\rho_0$ C/m $(-d/2 < z < 0)$ where ρ_0 is a positive constant. (*a*) Find the electric field intensity **E** everywhere in the xy plane, expressing your result as a function of cylindrical radius ρ. (*b*) Simplify your part *a* result for the case in which radius $\rho >> d$, and express this result in terms of charge $q = \rho_0 d/2$.

2.18 (a) Find **E** in the plane $z = 0$ that is produced by a uniform line charge, ρ_L, extending along the z axis over the range $-L < z < L$ in a cylindrical coordinate system. (b) If the finite line charge is approximated by an infinite line charge $(L \to \infty)$, by what percentage is E_ρ in error if $\rho = 0.5L$? (*c*) Repeat (*b*) with $\rho = 0.1L$.

2.19 A line having charge density $\rho_0\,|z|$ C/m and of length ℓ is oriented along the z axis at $-\ell/2 < z < \ell/2$. (*a*) Find the electric field intensity **E** everywhere in the xy plane, expressing your result in cylindrical coordinates. (b) Evaluate your part *a* result in the limit as L approaches infinity.

2.20 A line charge of uniform charge density ρ_0 C/m and of length ℓ is oriented along the z axis at $-\ell/2 < z < \ell/2$. (*a*) Find the electric field strength, **E**, in magnitude and direction at any position along the x axis. (*b*) With the given line charge in position, find the force acting on an identical line charge that is oriented along the x axis at $\ell/2 < x < 3\ell/2$.

2.21 A charged filament forms a circle of radius a in the xy plane with its center at the origin. The filament carries uniform line charge density $+\rho_0$ C/m for $-\pi/2 < \phi < \pi/2$ and $-\rho_0$ C/m for $\pi/2 < \phi < 3\pi/2$. Find the electric field intensity **E** at the origin.

2.22 Two identical uniform sheet charges with $\rho_s = 100$ nC/m^2 are located in free space at $z = \pm 2.0$ cm. What force per unit area does each sheet exert on the other?

2.23 A disk of radius a in the xy plane carries surface charge of density $\rho_s = \rho_{s0}/\rho$ C/m^2 where ρ_{s0} is a constant. Find the electric field intensity **E** everywhere on the z axis.

2.24 (a) Find the electric field on the z axis produced by an annular ring of uniform surface charge density ρ_s in free space. The ring occupies the region $z = 0$, $a \leq \rho \leq b$, $0 \leq \phi \leq 2\pi$ in cylindrical coordinates. (b) From your part a result, obtain the field of an infinite uniform sheet charge by taking appropriate limits.

2.25 A disk of radius a in the xy plane carries surface charge of density $\rho_{s1} = +\rho_{s0}/\rho$ C/m^2 for $0 < \phi < \pi$, and $\rho_{s2} = -\rho_{s0}/\rho$ C/m^2 for $\pi < \phi < 2\pi$, where ρ_{s0} is a constant. (a) Find the electric field intensity **E** everywhere on the z axis. (b) Specialize your part a result for distances $z >> a$.

2.26 (a) Find the electric field intensity on the z axis produced by a cone surface that carries charge density $\rho_s(r) = \rho_0/r$ C/m^2 in free space. The cone has its vertex at the origin and occupies the region $\theta = \alpha$, $0 < r < a$, and $0 < \phi < 2\pi$ in spherical coordinates. Differential area for a cone is given in spherical coordinates as $da = r \sin\alpha \, dr \, d\phi$. ($b$) Find the total charge on the cone. (c) Specialize your result of part a to the case in which $\alpha = 90°$, at which the cone flattens to a disk in the xy plane. Compare this result to the answer to problem 2.23. (d) Show that your part a result becomes a point charge field when $z >> a$. (e) Show that your part a result becomes an inverse-z-dependent **E** field when $z << a$.

2.27 Given the electric field $\mathbf{E} = (4x - 2y)\mathbf{a}_x - (2x + 4y)\mathbf{a}_y$, find ($a$) the equation of the streamline that passes through the point $P(2, 3, -4)$; (b) a unit vector specifying the direction of **E** at $Q(3, -2, 5)$.

2.28 An electric dipole (introduced in Problem 2.7, and discussed in detail in Section 4.7) consists of two point charges of equal and opposite magnitude $\pm q$ spaced by distance d. With the charges along the z axis at positions $z = \pm d/2$ (with the positive charge at the positive z location), the electric field in spherical coordinates is given by $\mathbf{E}(r, \theta) = [qd/(4\pi\epsilon_0 r^3)][2\cos\theta\,\mathbf{a}_r + \sin\theta\,\mathbf{a}_\theta]$, where $r >> d$. Using rectangular coordinates, determine expressions for the vector force on a point charge of magnitude q (a) at $(0, 0, z)$; (b) at $(0, y, 0)$.

2.29 If $\mathbf{E} = 20e^{-5y}(\cos 5x\mathbf{a}_x - \sin 5x\mathbf{a}_y)$, find ($a$) $|\mathbf{E}|$ at $P(\pi/6, 0.1, 2)$; (b) a unit vector in the direction of **E** at P; (c) the equation of the direction line passing through P.

2.30 For fields that do not vary with z in cylindrical coordinates, the equations of the streamlines are obtained by solving the differential equation $E_p/E_\phi = d\rho/(\rho d\phi)$. Find the equation of the line passing through the point $(2, 30°, 0)$ for the field $\mathbf{E} = \rho\cos 2\phi\,\mathbf{a}_\rho - \rho\sin 2\phi\mathbf{a}_\phi$.